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Stokes flow past a sphere coated with a thin fluid film

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The present study examines the steady, axisymmetric Stokes flow past a sphere coated with a thin, immiscible fluid layer. Inertial effects are neglected for both the outer fluid and the fluid film, and surface tension forces are assumed large compared with the viscous forces which deform the fluid film. Furthermore, the present analysis assumes that the mechanism driving the fluid circulation within the film is not too large. From force equilibrium on the film we find that a steady fluid film can only partially cover the sphere, i.e. the film must be held to the sphere by surface tension forces at the contact line. The extent of the sphere covered by the film is specified, in terms of the solid–fluid contact angle, by the condition of global force equilibrium on the fluid film.

Using a perturbation scheme based on the thinness of the fluid layer the solution to the flow field is obtained analytically, except for the fluid-film profile (i.e. the fluid–fluid interface) which requires numerical calculations. One of the principal results is an expression for the drag force on the fluid-coated particle. In particular, we find that the drag on a sphere is reduced by the presence of a fluid coating when the ratio of the film fluid viscosity to the surrounding fluid viscosity is less than $\frac{1}{4}$. Detailed numerical computations are conducted for a few typical cases. The calculations show that a film of prescribed areal extent, i.e. specified contact angle, is only possible when the magnitude of the driving force on the film is below some maximum value. A simple experiment was also performed, and photographs, which qualitatively illustrate the fundamental fluid-film configurations predicted by the theory, are presented.

1. Introduction

The flow due to a spherical body moving in a viscous, incompressible fluid at small Reynolds number has been a problem of both practical and theoretical importance. The translation of a rigid sphere was first considered by Stokes (1851), motivated by his interest in the effects of fluid friction on the motion of pendulums. A considerable time later Hadamard (1911) examined the translational motion of a spherical drop of immiscible fluid in order to determine the effect of internal fluid circulation on the drag force. Basset (1888) first considered the translation of a solid sphere when the fluid is allowed to slip at the sphere's surface. Basset assumed that the tangential component of the fluid velocity at the solid surface was proportional to the local tangential shear stress. Countless other works have considered the motion of rigid and fluid spheres in a variety of primary flows and bounded domains. Here we will consider a variation of the classical unbounded problem by examining the incompressible low-Reynolds-number flow past a solid sphere having a thin layer of immiscible fluid covering its surface, i.e. a fluid film.

In the present analysis the Reynolds number based on the free-stream velocity U_0, the outer fluid's viscosity μ' and density ρ', and the particle radius R_0 will be assumed small, with the corresponding inertial effects neglected. The mechanism driving the fluid-film circulation will be assumed sufficiently weak so that the fluid velocity in the film is small compared with the free-stream velocity (the condition which must be satisfied for this to be true will be discussed shortly). Furthermore, an appropriate Reynolds number for the fluid film will be assumed small enough so that the inertial effects may be neglected. Attention is restricted to axisymmetric flows in which the characteristic film thickness h_0 is small compared with the radius R_0 of the solid sphere. The fluid–fluid interface will have a radius specified by $\bar{R} = R_0(1+\epsilon f(\theta))$, where the film thinness parameter $\epsilon = h_0/R_0 \ll 1$, θ is the angle between a field point and the symmetry axis, and $f(\theta)$ is an unknown function to be determined as part of the solution. Since the fluid–fluid interface differs only slightly from a sphere, we require the surface tension forces to be large compared with the viscous forces which tend to deform the fluid film. In particular, we take $\beta = \mu U_0/\alpha \ll 1$ and $\beta' = \mu' U_0/\alpha \ll 1$, where α is the constant surface tension and μ the viscosity of the film fluid. In addition, the effect of intermolecular forces on the fluid film will be neglected.

A straightforward perturbation expansion of the flow field in terms of the thinness parameter ϵ will be used to construct the solution. We will see that this procedure essentially involves determining an order ϵ correction to the uniform flow past a solid sphere, due to the presence of the thin fluid film.

Before beginning the analysis a few qualitative features of the problem can be anticipated by considering the dimensional characteristics of the flow field. Firstly, the magnitude of the stresses in the outer flow field will be of order $\mu' U_0/R_0$ whereas the shear stress within the fluid film will be of order $\mu u/h_0$, where u represents the magnitude of the velocity in the film. Since the shear stress must be continuous at the fluid–fluid interface, we must have $u = O(\epsilon\mu' U_0/\mu)$. Consequently, u will be small provided the viscosity of the outer fluid, μ', is not too large, i.e. the driving force is sufficiently weak. Furthermore, within the fluid film there will be a constant pressure of magnitude $2\alpha/R_0$, since the fluid–fluid interface is spherical to first order and, due to the thinness of the fluid layer, the pressure generated within the fluid film by the outer flow will be large and of the order $\mu R_0 u/h_0^2 = \mu' U_0/\epsilon R_0$. Note that the large pressure generated in the film occurs for the same reason as does the large pressure in classical lubrication theory. From the magnitude of the stresses present we can obtain information concerning the fluid-film configurations which may occur. If we consider global force equilibrium for a fluid film which we assume completely covers the solid sphere ($f(\theta) > 0$, $0 \leqslant \theta \leqslant \pi$, i.e. there are no contact lines where the film vanishes), we find that such a steady fluid-film configuration is not possible. This is because the relatively large pressure of $O(\mu' U_0/\epsilon R_0)$ within the film results in a net force on the fluid film which cannot be balanced by the other, relatively smaller, stresses present. Note that the constant pressure term of $O(\alpha/R_0)$ is also large since $\beta \ll 1$, but contributes no net force on the film when it covers the entire sphere. Consequently, in order to have a steady flow situation it is necessary for the fluid film to only partially cover the solid sphere, so that force equilibrium can be established through the surface tension force at the contact line, i.e. the fluid film must be held to the sphere by the surface tension force at the contact line. From this we can anticipate the following basic axisymmetric fluid film configurations: (1) a fluid film which has two contact lines at $\theta = \theta_0$ and θ_1 with no

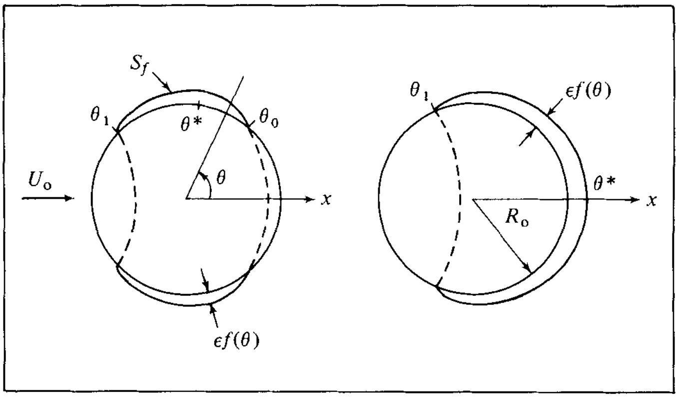


FIGURE 1. Fluid-film geometry. The left-hand side depicts a film with two contact lines and the right-hand side depicts a film with a single contact line. θ_0 and θ_1 are the positions of the fluid-film contact lines, and θ^* denotes the location of the maximum film thickness.

fluid film present over a portion of the front and rear of the sphere, i.e. a belt of fluid around the sphere, and (2) a fluid film which has a single contact line at either $\theta = \theta_1$ or θ_0 and therefore covers either the rear or front of the sphere (see figure 1).

The study begins in § 2 with a presentation of the governing equations and boundary conditions, followed by a formulation of the problem using expansions of the flow field in terms of the thinness parameter ϵ. In § 3 the solution is constructed and the correction to the drag force, due to the presence of the fluid film, is determined. A few illustrative examples are considered in § 4 along with the results of a simple experiment.

2. Governing equations and boundary conditions

Since the Reynolds number $Re = \rho' U_0 R_0/\mu'$ has been assumed small, the outer fluid velocity **U** and pressure P satisfy the Stokes equations. In non-dimensional form we have

$$\nabla^2 \mathbf{U} = \nabla P, \quad \nabla . \mathbf{U} = 0, \tag{1}$$

where the dimensionless velocity, pressure and spatial co-ordinates are based on U_0, $\mu' U_0/R_0$, and R_0 respectively (in the sequel dimensionless quantities will be used unless otherwise stated).

In the fluid film the velocity **u** and pressure p also satisfy the Stokes equations

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla . \mathbf{u} = 0, \tag{2}$$

where the dimensionless pressure is based on $\mu U_0/R_0$, and the dimensionless velocity and spatial co-ordinates are based on U_0 and R_0 as before. The specific Reynolds number which must be small to justify the neglect of the inertial terms will be made clear after the solution is obtained.

In formulating the problem we take the symmetry axis to be the x axis and introduce the spherical polar co-ordinates (R, θ, ϕ), where θ is the angle between a field point and the symmetry axis. The components of the velocity field in spherical co-ordinates for the outer fluid and the fluid film are respectively $(U, V, 0)$ and $(u, v, 0)$.

The boundary conditions for the outer fluid far from the sphere are

$$\mathbf{U} \rightarrow \mathbf{e}_x \quad \text{and} \quad P \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty, \tag{3}$$

where $\mathbf{e}_x$ is the unit vector in the x direction. On the solid sphere $R = 1$ the velocity either in the fluid film or the outer fluid, whichever one is in contact with the sphere, must satisfy the no-slip condition. At the fluid–fluid interface (S_f) we must have: (1) continuity of the tangential component of velocity, (2) no fluid flux across the interface, (3) continuity of the tangential stress, and (4) the discontinuity in the normal stress must be equal to the product of the surface tension α and the sum of the principal curvatures of the surface κ. Accordingly, on $R = 1+\epsilon f(\theta)$ we require

$$\mathbf{u}.\mathbf{t} = \mathbf{U}.\mathbf{t}, \quad \mathbf{u}.\mathbf{n} = \mathbf{U}.\mathbf{n} = 0, \tag{4),(5}$$

$$\tau_{nt} = \frac{\mu'}{\mu} T_{nt}, \quad \beta\left(\tau_{nn} - \frac{\mu'}{\mu} T_{nn}\right) = -\kappa, \tag{6),(7}$$

where $\mathbf{t}$ and $\mathbf{n}$ are respectively the unit tangent and normal vectors to S_f, T_{nt} and T_{nn} are respectively the outer fluid's dimensionless stresses (based on $\mu' U_0/R_0$) in the directions tangent and normal to S_f, τ_{nt} and τ_{nn} are similarly the fluid film's dimensionless stresses (based on $\mu U_0/R_0$), and as defined earlier $\beta = \mu U_0/\alpha$.

Although at this point the problem appears difficult from an analytical viewpoint, the fact that the fluid film is thin enables considerable simplification. Firstly, for a thin fluid film we anticipate that there is a leading-order outer flow driving fluid circulation within the film, which in turn leads to modifications of the outer flow. Consequently we assume

$$U = U^{(0)} + \epsilon U^{(1)} + \ldots, \quad V = V^{(0)} + \epsilon V^{(1)} + \ldots, \quad P = P^{(0)} + \epsilon P^{(1)} + \ldots, \tag{8}$$

with the governing equations for each order clearly being the Stokes equations. Secondly, since the surface tension forces are assumed large compared to the deforming effect of the stresses on the fluid–fluid interface it follows that the slope of the interface will be small. Note, however, that there will be a small region of non-uniformity near the contact line where the depth of the fluid film vanishes and the slope must equal $\tan\Phi$, where Φ is the contact angle. A consequence of the small slope of the interface is the fact that in the major portion of the film (excluding a small neighbourhood of the contact line) the velocity in the film perpendicular to the solid surface, i.e. the radial component, will be small compared with the tangential component. In addition, since the fluid-film velocity is assumed small compared with the free-stream velocity U_0, we take

$$v = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \ldots, \tag{9}$$

$$u = \epsilon^2 u^{(1)} + \epsilon^3 u^{(2)} + \ldots. \tag{10}$$

Furthermore, due to the thinness of the film the radial velocity gradients will be large compared to the tangential ones. Therefore, in order for the pressure gradient to balance the viscous stress term in the equation of motion we take

$$p = \frac{2}{\beta} + \frac{1}{\epsilon} p^{(1)} + p^{(2)} + \ldots, \tag{11}$$

where, as already mentioned, the first term in (11), $2/\beta$, is the constant pressure necessary so that the fluid–fluid interface is spherical to leading order (note equation (7), where, to the leading order, $\kappa \simeq 2$).

Introducing the thin film variable $\xi = (R-1)/\epsilon$, the equations (2) governing the fluid motion within the film become, to leading order,

$$\frac{\partial^2 v^{(1)}}{\partial \xi^2} = \frac{\partial p^{(1)}}{\partial \theta}, \quad \frac{\partial p^{(1)}}{\partial \xi} = 0, \tag{12),(13}$$

$$\frac{\partial u^{(1)}}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v^{(1)}) = 0. \tag{14}$$

For use in expressing the boundary conditions at the fluid–fluid interface $R = 1+\epsilon f(\theta)$ we have the unit normal and tangent vectors to S_f given by

$$\mathbf{n} \simeq \mathbf{e}_R - \epsilon\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + O(\epsilon^2), \tag{15}$$

$$\mathbf{t} \simeq \mathbf{e}_\theta + \epsilon\frac{\partial f}{\partial \theta}\mathbf{e}_R + O(\epsilon^2), \tag{16}$$

where $(\mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi)$ are the the unit vectors in spherical polar co-ordinates. Furthermore, since $\kappa = \nabla . \mathbf{n}$ we have

$$\kappa \simeq 2 - \epsilon\left(\frac{\partial^2 f}{\partial \theta^2} + \cot\theta\frac{\partial f}{\partial \theta} + 2f\right) + O(\epsilon^2). \tag{17}$$

Therefore, introducing the components of stress in spherical co-ordinates, namely τ_{RR}, $\tau_{R\theta}$, T_{RR} and $T_{R\theta}$, the boundary conditions on $R = 1+\epsilon f(\theta)$ become

$$v - V + \epsilon\frac{\partial f}{\partial \theta}(u-U) + O(\epsilon^2) \simeq 0, \quad u - \epsilon\frac{\partial f}{\partial \theta}v + O(\epsilon^2) \simeq 0,$$

$$U - \epsilon\frac{\partial f}{\partial \theta}V + O(\epsilon^2) \simeq 0, \quad \tau_{R\theta} - \frac{\mu'}{\mu}T_{R\theta} + O(\epsilon) \simeq 0,$$

$$\beta\left[\tau_{RR} - \frac{\mu'}{\mu}T_{RR} + O(\epsilon)\right] \simeq -\kappa.$$

These can be further approximated using a Taylor series expansion of the outer fluid's velocity and stresses about $R = 1$, giving

$$v - V + \epsilon\left[\frac{\partial f}{\partial \theta}(u-U) - f\frac{\partial V}{\partial R}\right] \simeq 0, \tag{18}$$

$$u - \epsilon\frac{\partial f}{\partial \theta}v \simeq 0, \tag{19}$$

$$U + \epsilon\left[f\frac{\partial U}{\partial R} - \frac{\partial f}{\partial \theta}V\right] \simeq 0, \tag{20}$$

$$\tau_{R\theta} - \frac{\mu'}{\mu}T_{R\theta} \simeq 0, \tag{21}$$

$$\beta\left[\tau_{RR} - \frac{\mu'}{\mu}T_{RR}\right] \simeq -\kappa, \tag{22}$$

where it is understood that the variables associated with the outer flow field are evaluated at $R = 1$, whereas the fluid-film flow field is evaluated at $\xi = f(\theta)$.

Substituting the expansions of the velocity fields (8), (9) and (10) into (18), (19) and (20) gives

$$V^{(0)} - \epsilon\left[v^{(1)} - V^{(1)} - f\frac{\partial V^{(0)}}{\partial R} - \frac{\partial f}{\partial \theta}U^{(0)}\right] \simeq 0, \tag{23}$$

$$u^{(1)} - \frac{\partial f}{\partial \theta}v^{(1)} \simeq 0, \tag{24}$$

$$U^{(0)} + \epsilon\left[U^{(1)} + f\frac{\partial U^{(0)}}{\partial R} - \frac{\partial f}{\partial \theta}V^{(0)}\right] \simeq 0. \tag{25}$$

For use in the stress conditions (21) and (22) we evaluate the stress components giving

$$\tau_{R\theta} = R\frac{\partial}{\partial R}\left(\frac{v}{R}\right) + \frac{1}{R}\frac{\partial u}{\partial \theta} \simeq \frac{\partial v^{(1)}}{\partial \xi} + O(\epsilon), \tag{26}$$

$$\tau_{RR} = -p + 2\frac{\partial u}{\partial R} \simeq -\frac{2}{\beta} - \frac{1}{\epsilon}p^{(1)} + O(1), \tag{27}$$

$$T_{R\theta} = \frac{\partial V^{(0)}}{\partial R} - \frac{V^{(0)}}{R} + \frac{1}{R}\frac{\partial U^{(0)}}{\partial \theta} + O(\epsilon) = T^{(0)}_{R\theta} + O(\epsilon), \tag{28}$$

$$T_{RR} = -P^{(0)} + 2\frac{\partial U^{(0)}}{\partial R} + O(\epsilon) = T^{(0)}_{RR} + O(\epsilon). \tag{29}$$

Consequently, using (17), (26), (27), (28) and (29), the boundary conditions (21) and (22) on the stresses become

$$\frac{\partial v^{(1)}}{\partial \xi} = \frac{\mu'}{\mu}T^{(0)}_{R\theta}(1, \theta), \tag{30}$$

$$-\frac{\beta}{\epsilon}p^{(1)} + O(\beta, \beta') \simeq \epsilon\left(\frac{\partial^2 f}{\partial \theta^2} + \cot\theta\frac{\partial f}{\partial \theta} + 2f\right) + O(\epsilon^2). \tag{31}$$

The restriction that β and β' are small becomes specific at this point, namely, in order that the left-hand and right-hand sides of (31) are of $O(\epsilon)$ with the error of higher order, we require $\beta = O(\epsilon^2)$ and $\beta' < O(\epsilon)$.

The formulation of the problem is now complete with the governing equations and boundary conditions leading to the following hierarchy of problems.

The leading-order outer flow problem is

$$\nabla^2\mathbf{U}^{(0)} = \nabla P^{(0)}, \quad \nabla . \mathbf{U}^{(0)} = 0 \tag{32}$$

with the boundary conditions

$$U^{(0)} = V^{(0)} = 0 \quad \text{on} \quad R = 1, \tag{33}$$

$$\mathbf{U}^{(0)} \to \mathbf{e}_x \quad \text{and} \quad P^{(0)} \to 0 \quad \text{as} \quad R \to \infty. \tag{34}$$

The leading-order fluid-film problem is

$$\frac{\partial^2 v^{(1)}}{\partial \xi^2} = \frac{\partial p^{(1)}}{\partial \theta}, \quad \frac{\partial p^{(1)}}{\partial \xi} = 0, \tag{35),(36}$$

$$\frac{\partial u^{(1)}}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v^{(1)}) = 0, \tag{37}$$

with the boundary conditions

$$v^{(1)} = u^{(1)} = 0 \quad \text{on} \quad \xi = 0, \tag{38}$$

$$\left.\begin{aligned} \frac{\partial v^{(1)}}{\partial \xi} &= \frac{\mu'}{\mu} T^{(0)}_{R\theta}(1,\theta) \quad (39) \\ u^{(1)} - \frac{\partial f}{\partial \theta} v^{(1)} &= 0 \quad (40) \end{aligned}\right\} \quad \text{on} \quad \xi = f(\theta),$$

where the film profile $f(\theta)$ is determined from (31), i.e.

$$\frac{\partial^2 f}{\partial \theta^2} + \cot\theta \frac{\partial f}{\partial \theta} + 2f = -\frac{\beta}{\epsilon^2} p^{(1)}, \tag{41}$$

with appropriate boundary conditions on $f(\theta)$ to be discussed in the following section.

The problem determining the first correction to the outer flow is

$$\nabla^2 \mathbf{U}^{(1)} = \nabla P^{(1)}, \quad \nabla . \mathbf{U}^{(1)} = 0, \tag{42}$$

with the following boundary conditions: for that portion of the sphere not covered by the film we have the no-slip condition,

$$U^{(1)} = V^{(1)} = 0 \quad \text{on} \quad R = 1, \quad 0 \leqslant \theta < \theta_0 \quad \text{and} \quad \theta_1 < \theta \leqslant \pi, \tag{43}$$

for that portion of the sphere covered by the film,

$$\left.\begin{aligned} U^{(1)} &= -f \frac{\partial U^{(0)}}{\partial R} \\ V^{(1)} &= v^{(1)} - f \frac{\partial V^{(0)}}{\partial R} \end{aligned}\right\} \quad \text{on} \quad R = 1, \quad \theta_0 \leqslant \theta \leqslant \theta_1, \tag{44}$$

where we have made use of the leading-order boundary conditions $U^{(0)} = V^{(0)} = 0$ on $R = 1$ and in (44) $v^{(1)}$ is evaluated on $\xi = f(\theta)$, and lastly, far from the sphere,

$$U^{(1)}, P^{(1)} \to 0 \quad \text{as} \quad R \to \infty. \tag{45}$$

3. Solution

3.1. *The leading-order outer flow solution*

From (32), (33) and (34) we see that the leading-order outer flow field corresponds to the uniform flow past a solid sphere. Using the singularity method for Stokes flow (Chwang & Wu 1975) the well-known solution of this problem is given by the sum of a uniform flow, a stokeslet, and a potential doublet, i.e.

$$\mathbf{U}^{(0)} = \mathbf{e}_x - \frac{3}{4}\left(\frac{\mathbf{e}_x}{R} + \frac{x\mathbf{x}}{R^3}\right) - \frac{1}{4}\left(\frac{\mathbf{e}_x}{R^3} - \frac{3x\mathbf{x}}{R^5}\right), \tag{46}$$

$$P^{(0)} = -\frac{3}{2}\frac{x}{R^3}, \tag{47}$$

where $R = |\mathbf{x}|$, $\mathbf{x}$ being a position vector to a field point having Cartesian co-ordinates (x, y, z).

3.2. *The leading-order thin film solution*

From (36) we see that the pressure $p^{(1)}$ is a function only of θ, and therefore (35), (38) and (39) give

$$v^{(1)} = \tfrac{1}{2}G(\theta)\,\xi(\xi-2f)+K(\theta)\,\xi, \tag{48}$$

where we define

$$G(\theta) = \frac{\partial p^{(1)}}{\partial \theta}, \tag{49}$$

$$K(\theta) = \frac{\mu'}{\mu}T^{(0)}_{R\theta}(1,\theta). \tag{50}$$

From the continuity equation (37), the no-slip condition (38), and the expression for $v^{(1)}$ we obtain

$$u^{(1)} = \tfrac{1}{2}\xi^2\left\{G\frac{\partial f}{\partial \theta}-\frac{\partial K}{\partial \theta}-K\cot\theta-(\tfrac{1}{3}\xi-f)\left(\frac{\partial G}{\partial \theta}+G\cot\theta\right)\right\}. \tag{51}$$

Furthermore, if we integrate the continuity equation (37) across the film thickness from $\xi = 0$ to $f(\theta)$ and use the remaining boundary condition (40), we obtain the condition

$$\int_0^f v^{(1)}(\xi)\,d\xi = 0. \tag{52}$$

This is the obvious physical constraint that for steady flow the net volume flux through a section of the fluid film must be identically zero. Satisfying this condition gives

$$G(\theta) = \frac{3}{2}\frac{K(\theta)}{f(\theta)}. \tag{53}$$

After using (50), (28) and (46), (53) becomes

$$\frac{\partial p^{(1)}}{\partial \theta} = -\frac{9}{4}\frac{\mu'}{\mu}\frac{\sin\theta}{f(\theta)}. \tag{54}$$

Integrating from a reference point θ^* to θ, we obtain the pressure within the fluid film

$$p^{(1)}(\theta) = p^* - \frac{9}{4}\frac{\mu'}{\mu}\int_{\theta^*}^{\theta}\frac{\sin\theta'}{f(\theta')}\,d\theta', \tag{55}$$

where $p^* = p^{(1)}(\theta^*)$. Note that p^* is an undetermined constant which will be found as a part of the solution.

We can now readily deduce the specific condition which must be satisfied if the inertial terms in the governing equation for the fluid film are to be neglected. The magnitude of the viscous and inertial effects are proportional to $(\mu U_0/R_0^2)\,\epsilon^{-2}\,\partial^2 v/\partial\xi^2$ and $(\rho U_0^2/R_0)\,u\epsilon^{-1}\,\partial v/\partial\xi$ respectively. Using the expressions for the velocity field (48) and (51) along with $v = \epsilon v^{(1)}$ and $u = \epsilon^2 u^{(1)}$, we find that the inertial terms are small if

$$\epsilon\frac{\mu'}{\mu}\frac{\rho U_0 h_0}{\mu} \ll 1. \tag{56}$$

Recalling that $\epsilon(\mu'/\mu)\,U_0$ is the magnitude of the fluid velocity within the film, we see that (56) requires the Reynolds number for the fluid film based on the film thickness h_0

to be small. Using the Reynolds number for the outer flow $Re = (\rho' U_0 R_0/\mu')$ it is useful to rewrite condition (56) as

$$\left(\epsilon \frac{\mu'}{\mu}\right)^2 \frac{\rho}{\rho'} Re \ll 1. \tag{57}$$

Since $\epsilon\mu'/\mu$ and Re have already been assumed small, this last expression clearly indicates the additional restrictions imposed on the properties of the two fluids by (56).

After using (55) the equation (41) determining the film profile $f(\theta)$ becomes

$$\frac{\partial^2 f}{\partial \theta^2} + \cot\theta \frac{\partial f}{\partial \theta} + 2f = -\frac{\beta}{\epsilon^2}\left[p^* - \frac{9}{4}\frac{\mu'}{\mu}\int_{\theta^*}^{\theta} \frac{\sin\theta'}{f(\theta')} d\theta'\right]. \tag{58}$$

Since the film thickness vanishes at the contact lines, we shall require

$$f \to 0 \quad \text{as} \quad \theta \to \theta_0, \theta_1. \tag{59}$$

Although for finite contact angles Φ the neighbourhood of the contact line is a region of non-uniformity, it is a general rule of the method of matched asymptotic expansions that the outer solution (in our case, the solution for $f(\theta)$ obtained from (58)) should be required to satisfy a boundary condition for the inner solution whenever possible (the inner solution referring to the solution valid in the neighbourhood of the contact line). Imposing an inner boundary condition on the outer solution then minimizes the modifications of the flow field needed by the inner solution, with the outer solution being determined to first order without considering the details of the inner solution. In a similar situation, Buckmaster (1977) made use of this fundamental asymptotic principle while studying the motion of viscous sheets down inclined surfaces.

An additional comment concerning the nature of the thin film approximation is appropriate at this point. The essence of the phenomena is that, due to the thinness of the fluid layer, large pressures are generated within the film by the outer flow; the pressure being inversely proportional to the film thickness. Therefore, as we can see from (54), as we approach the contact line ($f \to 0$) the pressure generated within the film becomes very large. Consequently, we can infer from this that near the contact line the fluid–fluid interface may not be nearly spherical in shape and therefore can satisfy the contact-angle condition for finite values of Φ, even though the viscous forces are small compared to the surface tension forces. The point is that a relatively weak outer flow generates very large pressures within the film near the contact line which result in leading order deviations in the fluid-film profile near the contact line.

In order to make condition (59) complete, it is necessary to determine the position of the contact lines θ_0 and θ_1 (or for a single contact line either θ_0 or θ_1). This can be done by constructing the solution that is valid near the contact line, i.e. the solution which satisfies the contact angle condition $\epsilon\,\partial f/\partial\theta = \tan\Phi$ at $\theta = \theta_0$ and θ_1 and matches to the outer solution. However, since the details of the flow in this inner region are of little interest, we can more easily obtain the positions θ_0 and θ_1 by considering global force equilibrium for the fluid film. The details are contained in the appendix. The principal result is that the position of one of the contact lines, either θ_0 or θ_1, is determined as a function of the solid–fluid contact angle Φ and the position of the other contact line. However, the position of one of the contact lines is not uniquely determined and consequently there exist many possible steady fluid-film configurations. In the case of a single contact line the problem is uniquely determined since specifying the absence of one of the contact lines is essentially the same as specifying its position. In an actual problem the occurrence of a particular fluid-film configuration is likely to be deduced from considerations of an appropriate initial-value problem describing the fluid-film development. This view was strongly supported by the experime s conducted.

Up to this point the problem has involved the thinness parameter ϵ, but has not directly involved the characteristic film thickness h_0. The remaining condition, which clearly determines h_0, is the specification of the volume of fluid within the fluid film. To leading order this becomes

$$\text{volume} = 2\pi h_0 R_0^2 \int_{\theta_0}^{\theta_1} f(\theta) \sin\theta \, d\theta. \tag{60}$$

In summary, we have obtained the fluid-film solution $v^{(1)}$, $u^{(1)}$ and $p^{(1)}$ as a function of the fluid-film thickness $f(\theta)$, which is determined from equation (58) with boundary conditions (59). The solution of equation (58) will be discussed in §4 along with a number of examples.

3.3. *The second-order outer flow solution*

The solution of the second order, outer Stokes flow problem with boundary conditions (43), (44) and (45) is most easily constructed using the Green's function method (Oseen 1927). In general, for unbounded flow about a surface S the Cartesian components of the velocity field $U_j = (U_1, U_2, U_3)$ at a field point $\mathbf{x}$, which vanish at infinity, are given, in terms of dimensional quantities, by

$$U_j(\mathbf{x}) = \frac{1}{8\pi\mu'} \int_S U_k \left[\mu' \frac{\partial \Gamma_{jk}}{\partial n} - P_j n_k\right] dS, \tag{61}$$

where within the integral U_k $(k = 1, 2, 3)$ are the components of the velocity field specified on S, n_k are the components of the unit outward normal to S, $\partial/\partial n$ is the derivative in the direction normal to S, and the Green's function $\Gamma_{jk}(\mathbf{x}, \boldsymbol{\xi})$ and $P_j(\mathbf{x}, \boldsymbol{\xi})$ are described as follows: $\Gamma_{jk}(\mathbf{x}, \boldsymbol{\xi})$ are the k components of the velocity field at $\boldsymbol{\xi}$ which satisfy $\Gamma_{jk} = 0$ for $\boldsymbol{\xi}$ on S, due to a point force located at $\boldsymbol{\xi} = \mathbf{x}$ and oriented in the j-direction with strength $8\pi\mu'$; $P_j(\mathbf{x}, \boldsymbol{\xi})$ is the corresponding pressure field at $\boldsymbol{\xi}$ due to the point force in the j-direction.

In our case the surface S in (61) is a sphere of radius $R = 1$. For that portion of the sphere covered by the film the velocity on S is given by (44). After utilizing the leading-order solution (46), (48), (53) and (50) (44) becomes

$$\left.\begin{aligned} U^{(1)} &= 0 \\ V^{(1)} &= \tfrac{3}{2} f(\theta) \sin\theta \left(1 - \frac{1}{4}\frac{\mu'}{\mu}\right) \end{aligned}\right\} \quad \text{on} \quad R = 1, \quad \theta_0 \leqslant \theta \leqslant \theta_1. \tag{62}$$

Since the velocity satisfies the no-slip condition (43) on that part of the sphere not covered by the film, the complete boundary condition on $R = 1$ becomes

$$\mathbf{U}^{(1)} = \begin{cases} V^{(1)}(\theta)\, \mathbf{e}_\theta = \tfrac{3}{2} f(\theta) \sin\theta \,(1 - \tfrac{1}{4}\mu'/\mu)\, \mathbf{e}_\theta & \text{for} \quad \theta_0 \leqslant \theta \leqslant \theta_1, \\ 0 \quad \text{for} \quad 0 \leqslant \theta < \theta_0 \quad \text{and} \quad \theta_1 < \theta \leqslant \pi. \end{cases} \tag{63}$$

Furthermore, since $\mathbf{n} = \mathbf{e}_R$ we have $\mathbf{U}^{(1)} . \mathbf{n} = 0$ on S and (61) becomes

$$U_j^{(1)}(\mathbf{x}) = \frac{1}{8\pi} \int_{\text{sphere}} U_k^{(1)} \frac{\partial \Gamma_{jk}}{\partial R} dS. \tag{64}$$

Although the Green's function Γ_{jk} has been given by Oseen (1927, p. 108), there is little practical value in computing the details of the velocity field from (64): especially since the Green's function involves considerable algebraic complexity. The quantity associated with the second-order outer flow which is of practical interest is the force exerted on the fluid or equivalently the force on the particle. This is most easily determined from the asymptotic form of the velocity field $U^{(1)}(\mathbf{x})$ as $\mathbf{x} \to \infty$. In this limit, the leading-order term to emerge will necessarily be the fundamental singularity, i.e. the stokeslet. From the singularity method we then have the well-known result that the force on the fluid is given by $8\pi\mu'$ times the stokeslet strength.

The asymptotic form of the Green's function $\Gamma_{jk}(\mathbf{x}, \boldsymbol{\xi})$ as $\mathbf{x} \to \infty$ is readily constructed directly or deduced from Oseen's result as

$$\Gamma_{jk}(\mathbf{x}, \boldsymbol{\xi}) \simeq \frac{\delta_{jk}}{r} + \frac{(\xi_j - x_j)(\xi_k - x_k)}{r^3} - \left(\frac{\delta_{jl}}{|\mathbf{x}|} + \frac{x_j x_l}{|\mathbf{x}|^3}\right)\left\{\frac{3}{4}\left(\frac{\delta_{kl}}{R} + \frac{\xi_k \xi_l}{R^3}\right) + \frac{1}{4}\left(\frac{\delta_{kl}}{R^3} - \frac{3\xi_k \xi_l}{R^5}\right)\right\}, \tag{65}$$

where $r = |\mathbf{x} - \boldsymbol{\xi}|$ and $R = |\boldsymbol{\xi}|$. The first term in (65) is clearly the unit point force at $\boldsymbol{\xi} = \mathbf{x}$ and the remaining terms are the leading-order terms of the image system within the sphere as $\mathbf{x} \to \infty$. In particular, the image system in (65) corresponds to a stokeslet and potential doublet located at the sphere centre. Substituting (65) and (63) into (64) we find, as $\mathbf{x} \to \infty$,

$$\mathbf{U}^{(1)} \simeq -\alpha^{(1)}\left(\frac{\mathbf{e}_x}{|\mathbf{x}|} + \frac{x\,\mathbf{x}}{|\mathbf{x}|^3}\right), \tag{66}$$

where

$$\alpha^{(1)} = \frac{3}{8}\int_{\theta_0}^{\theta_1} V^{(1)}(\theta)\sin^2\theta\, d\theta,$$

$$= \frac{9}{16}\left(1 - \frac{1}{4}\frac{\mu'}{\mu}\right)\int_{\theta_0}^{\theta_1} f(\theta)\sin^3\theta\, d\theta. \tag{67}$$

Equation (66) is clearly the velocity field associated with a stokeslet of strength $\alpha^{(1)}$ oriented in the negative x direction. The contribution to the force on the fluid due to the second-order outer flow is therefore given in dimensional terms by

$$-8\pi\mu' U_0 R_0 \epsilon \alpha^{(1)} \mathbf{e}_x.$$

3.4. *The drag force on the fluid-coated sphere*

The total force exerted on the outer fluid by the motion of the fluid-coated sphere is equal to $8\pi\mu'$ times the total stokeslet strength, i.e. the sum of the stokeslet strengths from the first- and second-order outer flow problem. Consequently, the drag on the fluid-coated particle is given in dimensional terms by

$$\text{drag} = 6\pi\mu' U_0 R_0\left[1 + \epsilon\frac{3}{4}\left(1 - \frac{1}{4}\frac{\mu'}{\mu}\right)D\right], \tag{68}$$

where

$$D = \int_{\theta_0}^{\theta_1} f(\theta)\sin^3\theta\, d\theta. \tag{69}$$

The first term in (68) is simply Stokes's law for the resistance of a solid sphere of radius R_0. We therefore see that drag reduction occurs when the sphere has a fluid film with a viscosity ratio $\mu/\mu' < \frac{1}{4}$. Furthermore, note that the two order-ϵ terms in (68) have the following physical interpretation: the first term corresponds to an increase in drag due to the increase in particle size associated with the presence of the fluid film, and the second term is the drag-reducing effect due to the fluid circulation within the film, i.e. the lubrication effect.

4. Results and discussion

In this section we will examine the following two fluid-film configurations, both of which were observed in a simple experiment: (1) a fluid film which has a single contact line at $\theta = \theta_1$ and covers the rear portion of the sphere and (2) a fluid film which has two contact lines and forms a belt around the sphere. The results for a film covering the front of the sphere with one contact line at $\theta = \theta_0$ are similar to (1) above and consequently will not be considered in detail. However, an additional characteristic typical of these configurations will be discussed. In each case the complete solution is given by the results obtained in §3, along with the film profile $f(\theta)$ which will be determined numerically from equation (58) with boundary conditions (59), i.e.

$$\frac{\partial^2 f}{\partial \theta^2} + \cot\theta \frac{\partial f}{\partial \theta} + 2f = -\frac{\beta}{\epsilon^2} p^* + k \int_{\theta^*}^{\theta} \frac{\sin\theta'}{f(\theta')} d\theta', \tag{70}$$

$$f \to 0 \quad \text{as} \quad \theta \to \theta_0, \theta_1, \tag{71}$$

where $k = \frac{9}{4}(\mu'/\mu)\beta/\epsilon^2$ and the position of the contact lines is known (as discussed earlier, this is equivalent to specifying the contact angle and one of the contact lines).

For convenience the characteristic film thickness h_0 is taken to be the maximum film thickness and we use that point as our reference location $\theta = \theta^*$ (see (55)). With this choice we must have

$$f = 1 \quad \text{and} \quad \frac{\partial f}{\partial \theta} = 0 \quad \text{at} \quad \theta = \theta^*. \tag{72}$$

In the case of two contact lines the position θ^* is not known *a priori* but is determined along with $(\beta/\epsilon^2)p^*$ for a specified value of k and specified contact line positions θ_0 and θ_1 (one of which is given in terms of the other, for prescribed Φ, by the condition of global force equilibrium). In the case of a single contact line with the rear portion of the sphere covered by the film the maximum film thickness is found to be located at $\theta^* = 0$ for all values of θ_1. This is because the sum of the principal curvatures κ is a decreasing function of θ for $0 \leqslant \theta \leqslant \pi$ (i.e. from (17) and (31)

$$\partial\kappa/\partial\theta = (\beta/\epsilon)\,\partial p^{(1)}/\partial\theta = -\tfrac{9}{4}(\mu'/\mu)\sin\theta/f(\theta)).$$

In terms of the numerical computations the film profile was constructed using an inverse procedure, namely, a value of θ^* was chosen and used as the initial point for a shooting-method scheme with initial conditions (72). For a specified value of k, various values of the parameter $(\beta/\epsilon^2)p^*$ were chosen (with θ^* fixed) until the condition $f = 0$ was satisfied at a prescribed location $\theta = \theta_1$. Then, in the case of two contact lines, the remaining portion of the profile for $\theta \leqslant \theta^*$ was computed and the value of θ_0, corresponding to the prescribed values of θ^* and $(\beta/\epsilon^2)p^*$, was determined. From a numerical standpoint it is actually more convenient to consider the equation obtained from (70) after differentiation with respect to θ, i.e.

$$\frac{\partial^3 f}{\partial \theta^3} + \cot\theta \frac{\partial^2 f}{\partial \theta^2} + \left(2 - \frac{1}{\sin^2\theta}\right)\frac{\partial f}{\partial \theta} = k\frac{\sin\theta}{f(\theta)}, \tag{73}$$

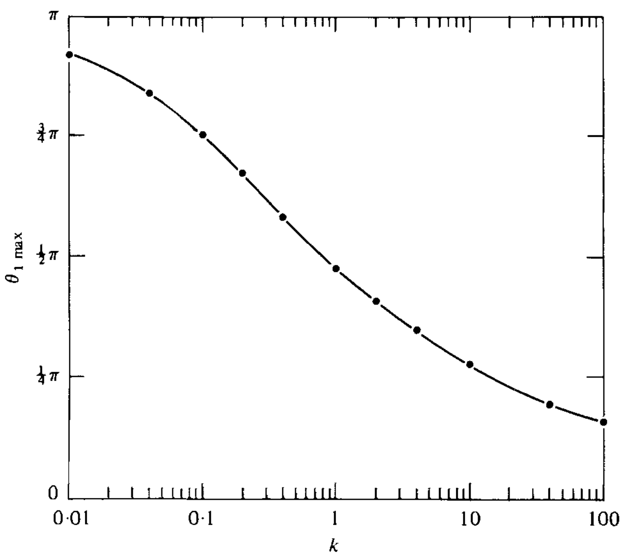


FIGURE 2. The maximum value of θ_1 as a function of the driving force parameter k for a film with a single contact line and $\theta^* = 0$.

where the additional initial condition required at $\theta = \theta^*$ for the shooting method is readily obtained from (70) and involves the parameter $(\beta/\epsilon^2)\,p^*$. The shooting method used to solve (73) employed a fourth-order Runge–Kutta scheme. For the discussion to follow, note that the parameter k is a measure of the driving force on the film since it involves the ratio (μ'/μ), and that the parameter $(\beta/\epsilon^2)\,p^*$ is a measure of the pressure in the film at $\theta = \theta^*$.

In the first example we consider a fluid film which has one contact line at $\theta = \theta_1$ and has its maximum depth at $\theta^* = 0$. Due to the singular nature of the coefficients in (73) at $\theta = 0$ the analytic solution valid near $\theta = 0$ was constructed and used in conjunction with the numerical calculation. In particular, near $\theta = 0, f(\theta)$ is easily found to be

$$f(\theta) \simeq 1 - A\theta^2 + \frac{1}{4}\left(\frac{k}{8} + \frac{A}{3}\right)\theta^4 + O(\theta^6), \tag{74}$$

where

$$A = \frac{1}{2}\left(1 + \frac{1}{2}\frac{\beta}{\epsilon^2}p^*\right).$$

The actual initial point for the numerical computation was taken to be $\theta = \Delta\theta \ll 1$ with conditions on f, $\partial f/\partial\theta$ and $\partial^2 f/\partial\theta^2$ at $\theta = \Delta\theta$ determined from (74).

In this case you find that, for specified values of k, the condition $f(\theta_1) = 0$ can only be satisfied for sufficiently large values of the pressure p^*. Furthermore, the position of the contact line θ_1 increases as p^* decreases and there exists a maximum value of θ_1 corresponding to a minimum value of p^*. For p^* less than the minimum value you find profiles $f(\theta)$ which decrease to some minimum value greater than zero and then diverge from the sphere, never satisfying $f = 0$. This behaviour is consistent with the fact that the sum of the principal curvatures κ decreases as θ increases, as already noted.

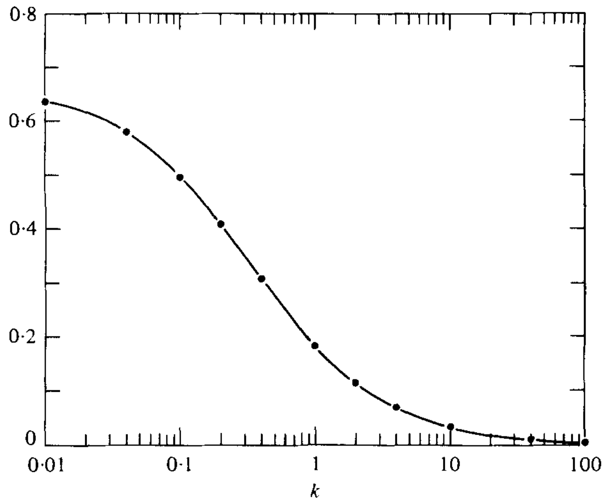


FIGURE 3. The drag-force factor D as a function of the driving force parameter k for a film with a single contact line at $\theta = \theta_{1\max}$ and $\theta^* = 0$.

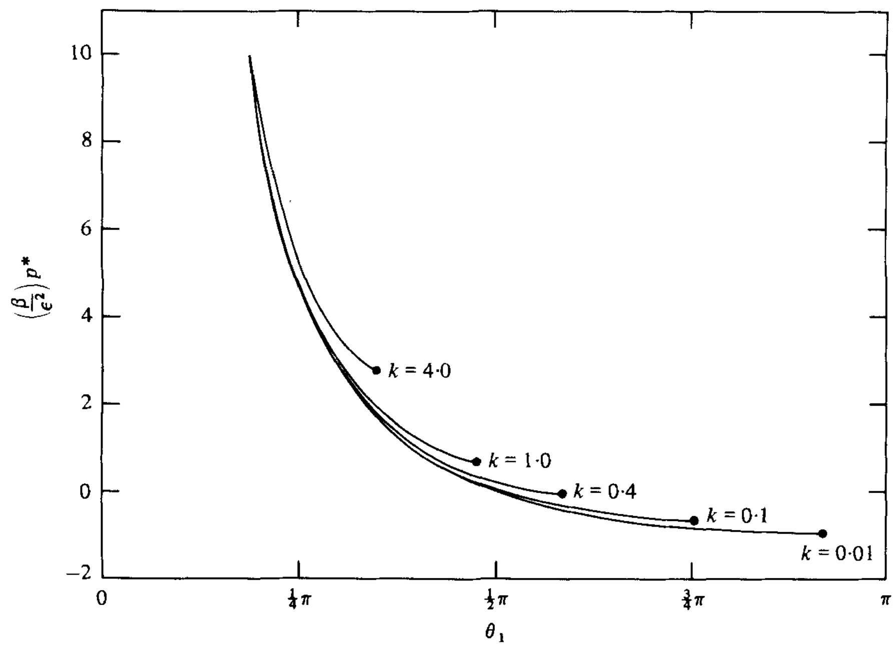


FIGURE 4. The pressure parameter $(\beta/\epsilon^2)p^*$ as a function of θ_1 for a film with a single contact line at $\theta = \theta_1$ and $\theta^* = 0$. Five values of the driving force parameter k are shown, and the largest value of θ_1 for each k is $\theta_{1\max}$.

Numerical calculations were completed for values of the driving-force parameter k between 0·01 and 100·0. Figure 2 is a plot of the maximum possible extent of the fluid film, i.e. $\theta_{1\max}$, versus the parameter k. Note that small values of k correspond to a weak driving force, and therefore as expected $\theta_{1\max}$ increases as k decreases. Furthermore, in figure 3 we see that, as the maximum extent of the film increases, the drag force factor D (equation 69) also increases. In figure 4 we have plotted $(\beta/\epsilon^2)p^*$ versus

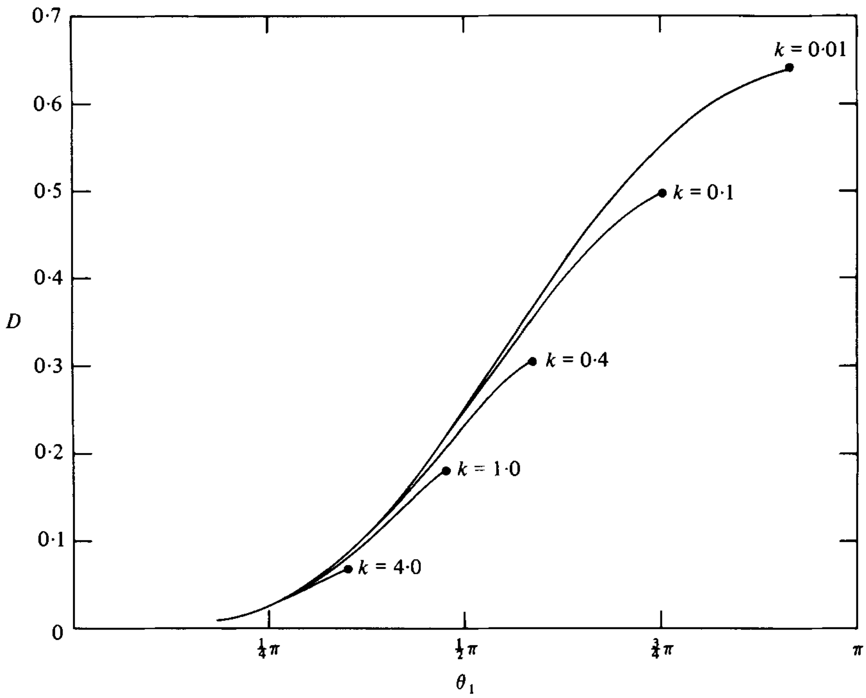


FIGURE 5. The drag force factor D as a function of θ_1 for a film with a single contact line at $\theta = \theta_1$ and $\theta^* = 0$. Five values of the driving-force parameter k are shown, and the largest value of θ_1 for each k is $\theta_{1\max}$.

θ_1 for $k = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}4$, $1{\cdot}0$ and $4{\cdot}0$. As already discussed, we observed that $(\beta/\epsilon^2)p^*$ decreases as θ_1 increases until the maximum value of θ_1 is reached, i.e. the largest value of θ_1 plotted in figure 4 for each k. In a specific problem, once the contact angle Φ is specified, the extent of the fluid-film θ_1 is determined by the condition of global force equilibrium discussed in the appendix. Therefore, for a specified value of the driving-force parameter k and contact angle Φ, the value of p^* is determined as part of the solution from figure 4; provided, of course, that the value of θ_1 for the given contact angle is a possible solution for the specified value of k, i.e. $\theta_1 \leqslant \theta_{1\max}$. In figure 5 the drag force factor D is plotted versus θ_1 for the same values of k shown in figure 4. We see that D increases as θ_1 increases, and in each case the maximum value of D corresponds to the maximum value of θ_1. Note that the drag force factor D is small for films with $\theta_1 < \frac{1}{2}\pi$. This is because the effect of the film on the drag force is only significant when the film is present near $\theta = \frac{1}{2}\pi$, where the shear stress exerted on the film by the outer fluid is a maximum. Consequently, for values of $k > 1{\cdot}0$, for which $\theta_{1\max} < \frac{1}{2}\pi$, the deviation in the drag force from that of a solid sphere alone is small.

From the velocity field $v = \epsilon v^{(1)}$ and $u = \epsilon^2 u^{(1)}$, where $v^{(1)}$ and $u^{(1)}$ are given by (48) and (51), the stream function for the fluid film is easily computed as

$$\psi(\xi, \theta) = \frac{3\mu'}{8\mu}\epsilon^2\xi^2\left(\frac{\xi}{f(\theta)} - 1\right)\sin^2\theta. \tag{75}$$

The streamlines for a typical case are sketched in figure 6.

A simple experiment was conducted in order to qualitatively observe the fluid-film

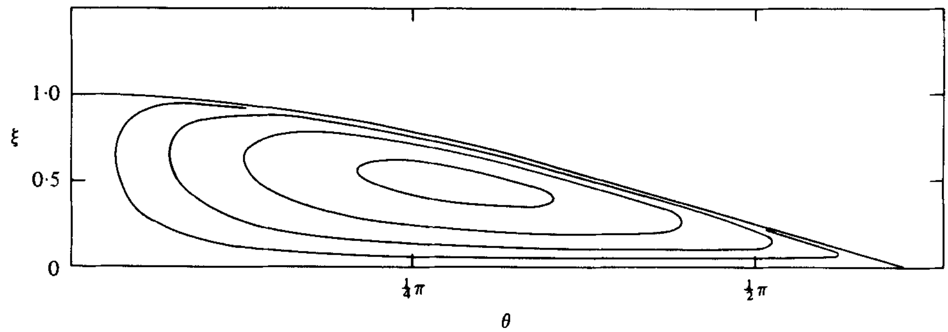
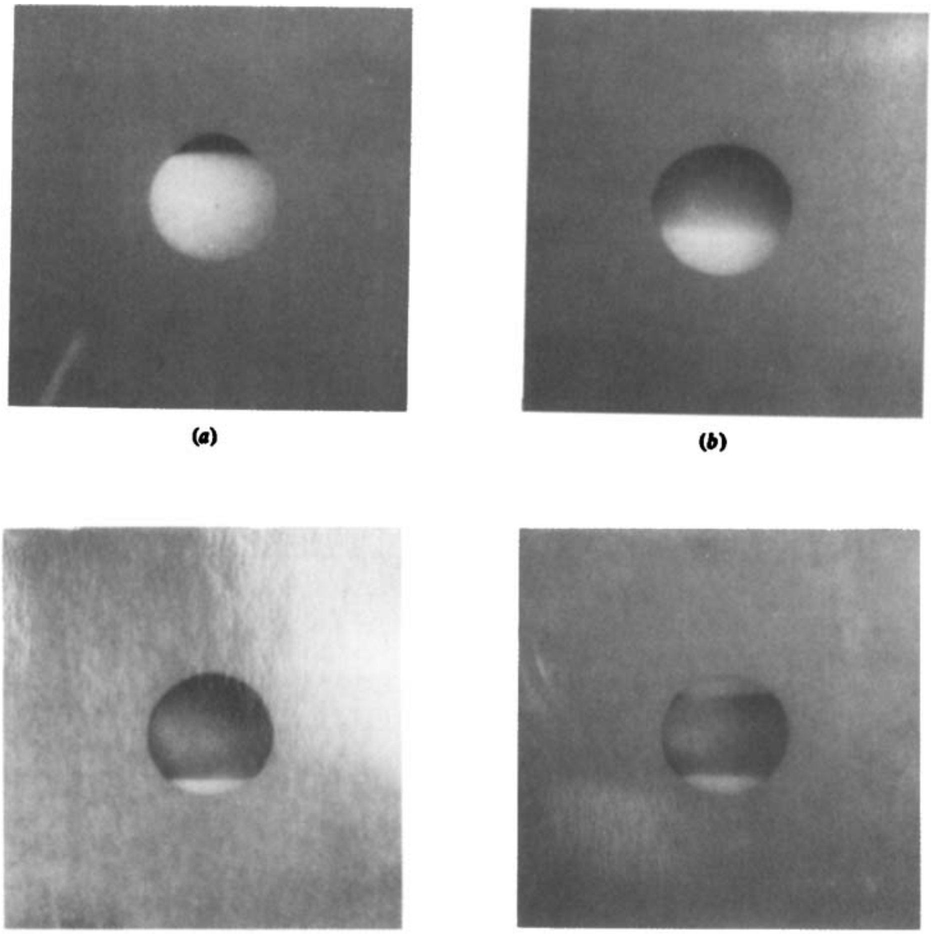


FIGURE 6. Streamlines for a typical fluid film having a single contact line and $\theta^* = 0$. $k = 0{\cdot}01$ and $(\beta/\epsilon^2)p^* = -0{\cdot}5$.

configurations. A plastic sphere (diameter = 6·35 mm) was sedimented in a tank containing a mixture of water and Pluracol V-10 (a liquid polyoxyalkylene) having a viscosity of approximately 1850 cP and a specific gravity of 1·07. At the shear rates of interest here, viscosity measurements using a Brookfield viscometer indicated that the Pluracol–water solution exhibited Newtonian rheological properties. The film fluid was a motor oil having a viscosity of 115 cP and a specific gravity of 0·90. The film viscosity was increased by mixing the motor oil with a small amount of a light-petroleum-based grease. The oil–grease mixtures, however, were prepared in such small quantities that viscosity measurements were not possible. The oil was slightly buoyant, but the ratio of buoyant forces to viscous forces in the experiment was very small. In the experiments conducted the Reynolds number, $Re = \rho' U_0 R_0/\mu'$, was within the range $1{\cdot}6 \times 10^{-3}$ to $1{\cdot}8 \times 10^{-3}$. The spheres were either fully or partially coated with motor oil and photographs of the sedimenting spheres were taken after all observable time-dependent development of the film had ceased. In figure 7, photographs of three different fluid films having a single contact line at $\theta = \theta_1$ are shown. The experiment supported the fact that increasing the film viscosity, i.e. decreasing k, enables the film to cover greater portions of the sphere. Figure 7 (*a*) shows the maximum extent of the film which could be obtained with motor oil alone. Increasing the viscosity of the oil film made it possible to increase the extent of the fluid film as shown in the figure 7 (*b*) and (*c*).

The second example to be considered is that of a fluid film forming a belt around the sphere. The behaviour of this film configuration will be illustrated by considering the details of a typical case. Here we will consider the case when $\theta_1 = 135°$, approximating the position of the forward contact line shown in the photograph of figure 7 (*d*). In the experiment a belt of an oil–grease mixture was initially coated on the sphere by rolling the sphere through a thin layer of the film fluid. After the fluid film reached its steady-state configuration during sedimentation photographs were taken. Figure 7 (*d*) was typical of the fluid films which could be generated in this fashion.

As already described, the numerical results in this case were obtained by determining the value of $(\beta/\epsilon^2)p^*$ which satisfied $f = 0$ at $\theta_1 = 135°$ for prescribed values of the maximum depth θ^* and the driving-force parameter k. After $(\beta/\epsilon^2)p^*$ was determined the remainder of the film profile for $\theta < \theta^*$ was computed, and the position of the rear contact line θ_0 was found. In figure 8 we have a plot of θ^* versus θ_0 for four values of k. For $k = 1{\cdot}0$, $4{\cdot}0$, and $10{\cdot}0$, the smallest value of θ_0 shown in each case corresponds to the smallest possible value of θ_0 capable of satisfying $f = 0$ at $\theta = \theta_1$.



(a) (b) (c) (d)

FIGURE 7. Photographs of three fluid films (a, b, c) which have a single contact line; the film covers the rear of the sphere and is absent over a portion of the front of the sphere, and a fluid film (d) with two contact lines, forming a belt around the sphere. The spheres are sedimenting towards the bottom of each photograph.

This physically means that for a specified magnitude of the driving force there is some maximum extent of the fluid film, i.e. a maximum value of $\theta_1 - \theta_0$. This is analogous to the maximum value of θ_1 found in the previous example. For $k = 0{\cdot}1$, as $\theta_0 \to 0$, θ^* is tending to zero, i.e. the present film configuration is tending to that of the one contact line case. This is consistent with results of the previous example. In the previous example when $k = 0{\cdot}1$, $\theta_{1\max}$ was slightly greater than 135° and therefore $\theta_1 = 135°$ with $\theta_0 = \theta^* = 0$ is an admissible solution. In figure 9 we have the drag-force factor D plotted against θ_0 for the same four values of k. Note that for $k = 0{\cdot}1$ and $1{\cdot}0$ the maximum value of D does not correspond to the maximum film extent (i.e. minimum θ_0). In this case, both the extent of the fluid film and the location of the maximum film thickness have an effect on the value of D. As the extent of the film increases, the location of the maximum film thickness moves toward the rear (figure 8, θ^* decreases as θ_0 decreases), and the film becomes thinner near $\theta = \frac{1}{2}\pi$. As the film becomes thinner

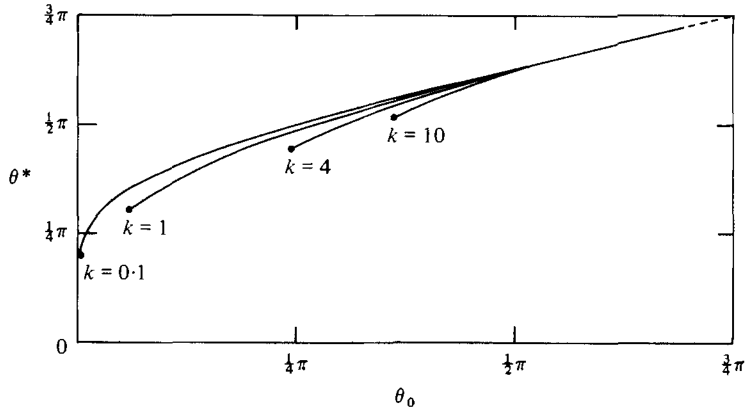


FIGURE 8. The location of the maximum film thickness θ^* as a function of the position of the rear contact line θ_0 for a film with two contact lines, $\theta = \theta_0$ and $\theta = \theta_1 = 135°$. Four values of the driving-force parameter k are shown.

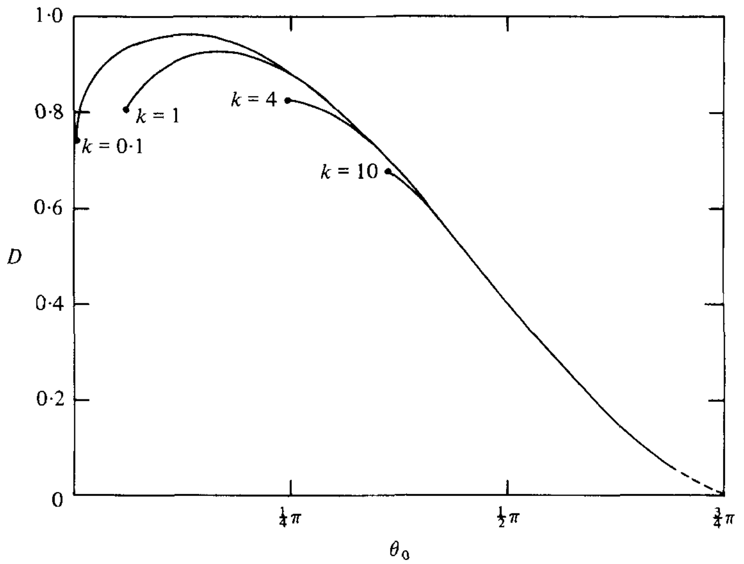


FIGURE 9. The drag-force factor D as a function of the position of the rear contact line θ_0 for a film with two contact lines, $\theta = \theta_0$ and $\theta = \theta_1 = 135°$. Four values of the driving-force parameter k are shown.

at $\theta = \frac{1}{2}\pi$ it becomes less effective in altering the drag force. Consequently, there is a value of θ_0 where the drag factor D is optimum. It is interesting to note that in both cases, $k = 0{\cdot}1$ and $1{\cdot}0$, the maximum value of D occurs when the maximum depth is at approximately the same position, namely, $\theta^* \simeq 76°$. The reason for this is not entirely clear. The streamlines for a typical situation are shown in figure 10.

The last comment concerns the fluid-film configurations which may occur when the film covers the forward portion of the sphere and has a single contact line at $\theta = \theta_0$. Although this case has much in common with the fluid films considered in the first example, there is one additional film profile which is unique to this situation. Since $\partial\kappa/\partial\theta = -\frac{9}{4}(\mu'/\mu)\sin\theta/f(\theta)$, the sum of the principal curvatures increases as you move

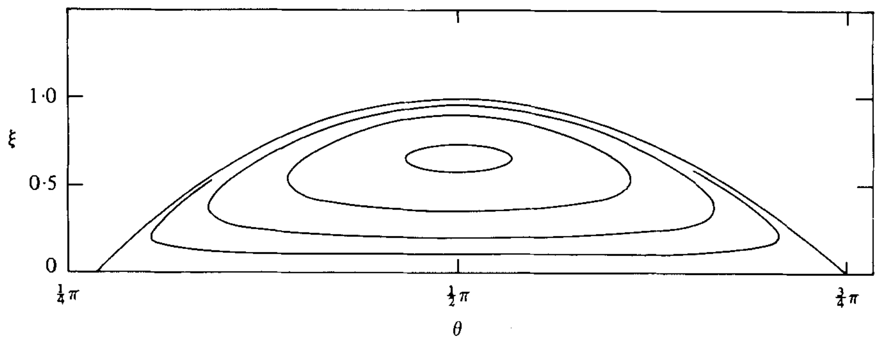


FIGURE 10. Streamlines for a typical fluid film which has two contact lines. $k = 1{\cdot}0$ and $(\beta/\epsilon^2)\,p^* = 1{\cdot}5$.

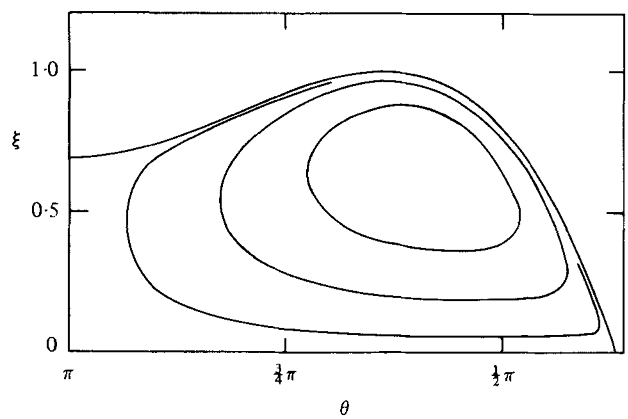


FIGURE 11. Streamlines for a typical fluid film which has a single contact line at $\theta = \theta_0$ and covers the front of the sphere with a local minimum thickness at $\theta = \pi$. $k = 10{\cdot}0$ and $(\beta/\epsilon^2)\,p^* = -5{\cdot}0$.

toward the rear of the sphere from $\theta = \pi$ to $\theta = 2\pi$. Consequently, here it is possible to have a film profile which has a local minimum at $\theta = \pi$, increases to a maximum at $\theta = \theta^* > \pi$, and then decreases to $f = 0$ at the rear contact line $\theta = \theta_0$. Such profiles were found for $k = 1{\cdot}0$ and $10{\cdot}0$, but were not obtained for smaller values of k, such as $k = 0{\cdot}1$. The streamline pattern for a typical case is shown in figure 11. The other film profiles found in this case, which were very similar to those of the first example, started at a maximum value at $\theta = \pi$ and decreased monotonically to $f = 0$ at $\theta = \theta_0$. In general, the role of the driving-force parameter k and the pressure parameter $(\beta/\epsilon^2)\,p^*$ were the same as before, and therefore further discussion will be omitted. In the experiments conducted it was unfortunately not possible to sediment the particle with a film covering the front of the sphere. The difficulty arose from the small buoyant force on the oil film at the front of the sphere which made such orientations unstable. Any small perturbation from a perfectly symmetric orientation of the particle resulted in rotation of the particle due to the torque generated by the buoyant force on the forward portion of the film.

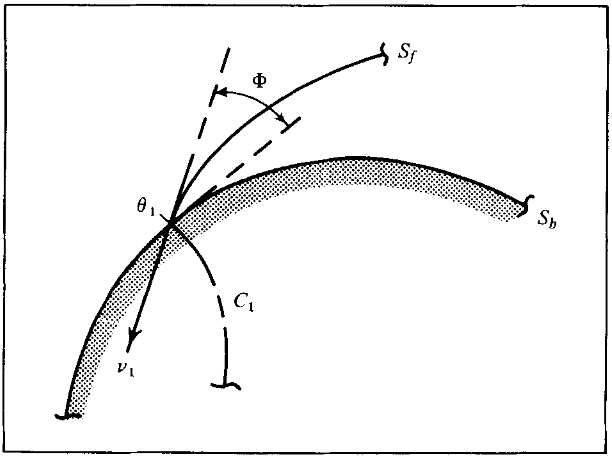


FIGURE 12. The film geometry near the contact line C_1 located at $\theta = \theta_1$. The geometry is similar near the contact line C_0 at $\theta = \theta_0$.

I wish to thank René Lara for making the excellent reproductions of the original photographs. This work was partially sponsored by the National Science Foundation.

Appendix

Since the sum of the forces acting on the fluid film must vanish we have, in dimensional quantities,

$$\int_{S_f} \mathbf{f}'\,dS + \int_{S_b} \mathbf{f}\,dS + \oint_{C_0} \alpha \boldsymbol{\nu}_0\,ds + \oint_{C_1} \alpha \boldsymbol{\nu}_1\,ds = 0, \tag{A 1}$$

where $\mathbf{f}'$ is the force per unit area at the fluid–fluid interface (S_f) due to the outer fluid, $\mathbf{f}$ is the force per unit area in the fluid film at the sphere surface (S_b) which is in contact with the film, and $\boldsymbol{\nu}_k$ ($k = 0, 1$) is a unit vector tangent to the surface S_f and perpendicular to the contact line C_k (see figure 12). Clearly, for a single contact line only one line integral would be present in (A 1).

Due to the symmetry of the flow under consideration only the sum of the force in the x-direction is non-trivial. Since the surface tension has been assumed constant, the line integrals are readily computed, giving, for the force balance in the x-direction,

$$\int_{S_f} f'_x\,dS + \int_{S_b} f_x\,dS = 2\pi R_0 \alpha[\sin\theta_1 \sin(\theta_1 + \Phi) - \sin\theta_0 \sin(\theta_0 - \Phi)], \tag{A 2}$$

where f'_x and f_x are the x-components of the tractions $\mathbf{f}'$ and $\mathbf{f}$, and the contact angle Φ is the angle between a tangent to S_f and a tangent to S_b measured from the fluid-film side of the contact line.

The leading-order terms on the left-hand side of (A 2) are due to the large fluid-film pressure. In particular, after neglecting terms of $O(\mu' U_0/R_0)$ and $O(\mu U_0/R_0)$ in f'_x and f_x we have

$$\frac{\mu U_0}{R_0}\int_0^{2\pi}\int_{\theta_0}^{\theta_1}\left[\frac{2}{\beta}+\frac{1}{\epsilon}p^{(1)}(\theta)\right]\cos\theta R_0^2 \sin\theta\,d\theta\,d\phi \simeq 2\pi R_0 \alpha[\sin\theta_1 \sin(\theta_1 + \Phi) - \sin\theta_0 \sin(\theta_0 - \Phi)], \tag{A 3}$$

where $p^{(1)}(\theta)$ is given by (55). Note that in obtaining (A 3) it is necessary to consider the behaviour of the solution valid near the contact line where the depth of the fluid film vanishes, since a singularity in the pressure and normal stress is expected. Very close to the contact line the flow field can be approximated by the flow near a sharp corner between two planes intersecting at an angle Φ (the contact angle). On one plane (the solid surface) the no-slip condition must be satisfied. On the other plane (the fluid–fluid interface) the normal component of velocity must vanish and the shear stress is prescribed. Since the details of this analysis are similar to considerations of Moffatt (1964) they will not be repeated. The primary result needed here is that the pressure and normal stress are logarithmically singular in the small inner region close to the contact line. Consequently, after integration their contribution to the total force on the fluid film is small compared to those terms retained in (A 3).

After partially integrating the left-hand side of (A 3), recalling $\beta = \mu U_0/\alpha$, and performing some simple manipulations, we obtain

$$2\sin\tfrac{1}{2}\Phi\sin(\theta_1+\theta_0)\cos(\theta_1-\theta_0+\tfrac{1}{2}\Phi)-(\beta/\epsilon)I = 0, \tag{A 4}$$

where

$$I = \int_{\theta_0}^{\theta_1} p^{(1)}(\theta)\cos\theta\sin\theta d\theta.$$

Equation (A 4) determines θ_1 for specified values of θ_0 and Φ or, alternatively, determines θ_0 for specified θ_1 and Φ. Since (A 4) determines the contact line up to a term of $O(\epsilon)$ we let $\theta_1 = \theta_1^{(0)}+\epsilon\theta_1^{(1)}+\dots$ (or alternatively we could let $\theta_0 = \theta_0^{(0)}+\epsilon\theta_0^{(1)}+\dots$) and find

$$2\sin\tfrac{1}{2}\Phi\{\sin(\theta_1^{(0)}+\theta_0)\cos(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi)+\epsilon\theta_1^{(1)}[\cos(\theta_1^{(0)}+\theta_0)\cos(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi) \\ -\sin(\theta_1^{(0)}+\theta_0)\sin(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi)]\}-(\beta/\epsilon)I+O(\epsilon^2) = 0. \tag{A 5}$$

For finite contact angles Φ we have from (A 5) the following hierarchy of equations for $\theta_1^{(0)}$ and $\theta_1^{(1)}$,

$$\sin(\theta_1^{(0)}+\theta_0)\cos(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi) = 0, \tag{A 6}$$

$$2\theta_1^{(1)}\sin\tfrac{1}{2}\Phi[\cos(\theta_1^{(0)}+\theta_0)\cos(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi)-\sin(\theta_1^{(0)}+\theta_0)\sin(\theta_1^{(0)}-\theta_0+\tfrac{1}{2}\Phi)] \\ -(\beta/\epsilon^2)I = 0. \tag{A 7}$$

The equations (A 6) and (A 7) have the two solutions

$$\left.\begin{aligned}\theta_1^{(0)} &= \theta_0+\tfrac{1}{2}(\pi-\Phi),\\ \theta_1^{(1)} &= -(\beta/2\epsilon^2)I[\sin(\tfrac{1}{2}\Phi)\cos(2\theta_0-\tfrac{1}{2}\Phi)]^{-1},\end{aligned}\right\} \tag{A 8}$$

and

$$\left.\begin{aligned}\theta_1^{(0)} &= \pi-\theta_0,\\ \theta_1^{(1)} &= (\beta/2\epsilon^2)I[\sin(\tfrac{1}{2}\Phi)\cos(2\theta_0-\tfrac{1}{2}\Phi)]^{-1}.\end{aligned}\right\} \tag{A 9}$$

If we set $\theta_0 = 0$ in the first solution (A 8) we obtain the special case of one contact line with the film absent from a portion of the front of the sphere. Similarly the single contact line with the film absent from the rear portion of the sphere is obtained using the alternative description $\theta_0 = \theta_0^{(0)}+\epsilon\theta_0^{(1)}$ in (A 4) and setting $\theta_1 = \pi$. In either case, since $0 < \Phi \leqslant \pi$, the maximum extent of a fluid film with one contact line and finite contact angle Φ is less than half of the sphere surface. Note that the second solution given by (A 9) describes the positions of two contact lines which, to first order, are located symmetrically about $\theta = \frac{1}{2}\pi$ and are independent of Φ. In this case the position of the contact lines are independent of Φ to first order because the leading-order contribution to the net force on the film is zero by symmetry.

For infinitesimal contact angles Φ the positions of the contact lines can also be determined from (A 5). However, in this case the solution for the film thickness $f(\theta)$ obtained from (58) is uniformly valid, since the small slope condition is satisfied everywhere. Consequently the fluid-film configuration $f(\theta)$ must satisfy the condition $\epsilon\, \partial f/\partial\theta = \tan\Phi$ at $\theta = \theta_0$ and θ_1 for specified small contact angles Φ. For calculation purposes this latter approach is more conveniently employed than the solution of (A 5). As will be illustrated in the examples, for small contact angles you find that it is possible to have a fluid film covering more than half of the sphere. We make this point since surfaces are rarely perfectly smooth and the observed contact angle (corresponding to Φ used here) could be small even though the actual contact angle is finite (Dussan V. 1979). In such cases it would also be possible to have two different observed contact angles Φ_0 and Φ_1, associated with the contact lines θ_0 and θ_1, further increasing the number of fluid-film configurations possible.

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